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LETTER TO THE EDITOR

Order reductions of Lorentz–Dirac-like equations

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Abstract. We discuss the phenomenon of pre-acceleration in the light of a method of successive approximations used to construct the physical order reduction of a large class of singular equations. A simple but illustrative physical example is analysed to gain more insight into the convergence properties of the method.

1. Introduction

In a recent paper [1] one of the present authors has proposed a numerical implementation of a method of successive approximations that allows automatic construction of the order reduction that contains the physical, non-runaway, solutions of a large class of singular differential equations, including the classical equation of motion of charged particles with radiation reaction [2] and fourth-order equations appearing in theories of gravitation with a quadratic Lagrangian [3] and in the study of quantum corrections to Einstein equations [4]. Apart from its practical interest, the convergence of the numerical method provides indirect but convincing evidence of the convergence of the analytic method.

The goal of this letter is twofold: we want to discuss the phenomenon of pre-acceleration in the frame of the method of successive approximations and to produce a more physical exact example in which the method can be analysed in full detail.

2. Pre-acceleration

Among the puzzling properties of the Lorentz–Dirac equation [2], which describes the motion of a radiating point charge, the pre-acceleration is one of the consequences of its singular structure. Let us consider the non-relativistic approximation to the Lorentz–Dirac equation (the so-called Abraham–Lorentz equation) in the case of a charge e that moves in a straight line under the action of an external force per unit mass $f(t)$:

$$\ddot{x} = f(t) + \tau_0 \dddot{x} \tag{1}$$

where

$$\tau_0 \equiv \frac{2e^2}{3mc^3}. \tag{2}$$

It is well known that if one eliminates the runaway solutions, the physical motion is described by the integrodifferential equation [1, 2, 5]

$$\ddot{x} = \int_0^\infty \mathrm{e}^{-u} f(t + \tau_0 u)\,\mathrm{d}u. \tag{3}$$

If the external force is $f(t) = f_0\delta(t)$ it is enough to insert this expression in equation (3) to obtain the order reduction

$$\ddot{x}(t) = \begin{cases} \dfrac{f_0}{\tau_0}\,\mathrm{e}^{t/\tau_0} & \text{if } t < 0 \\ 0 & \text{if } t > 0. \end{cases} \tag{4}$$

According to this the charge would start accelerating before the pulse reaches it.

This phenomenon has been widely discussed in connection with the smallness of τ_0 and taking into account the limitations of the classical theory, but we want to analyse it here from the point of view of the method of successive approximations [1], which starts from the approximation that neglects completely the radiation reaction

$$\ddot{x} = \Theta_0(t) \equiv f(t) \tag{5}$$

and iteratively constructs approximate reductions by substituting the previous approximation on the right-hand side of (1)

$$\ddot{x} = \Theta_{n+1}(t) \equiv f(t) + \tau_0\Theta_n'(t) = \sum_{k=0}^{n+1} \tau_0^k f^{(k)}(t). \tag{6}$$

Under the appropriate mathematical conditions, this method will converge to the exact reduction

$$\ddot{x} = \Theta(t) = \sum_{k=0}^{\infty} \tau_0^k f^{(k)}(t) \tag{7}$$

which is precisely the Taylor expansion of (3).

Now, one of the main hypotheses in the method of successive approximations is Bhabha's remark [6] that the physical solutions are precisely those that are regular in the limit $\tau_0 \to 0$, where according to (3) and (7) one recovers the second-order equation

$$\ddot{x} = f(t). \tag{8}$$

However, we can see that the pre-accelerated solution (4) is divergent in the limit $\tau_0 \to 0$, exactly as the remaining pathological solutions. In consequence, it cannot be constructed by the (analytical or numerical) method of successive approximations. One might see this as a limitation of the latter method, but we think that it is rather a limitation of the Abraham–Lorentz and similar equations.

To make clear our point of view, we will consider a pulse of small but non-null width. For simplicity we will take a Gaussian pulse,

$$f(t) = \frac{f_0}{\varepsilon\sqrt{\pi}}\,\mathrm{e}^{-(t/\varepsilon)^2} \tag{9}$$

but one could also consider any other pulse that recovers the value $f_0\delta(t)$ in the limit $\varepsilon \to 0$. After inserting (9) in (3) one gets the Newtonian equation of motion that contains the physical solutions:

$$\ddot{x} = \frac{f_0}{2\tau_0}\,\mathrm{e}^{t/\tau_0}\,\mathrm{e}^{\varepsilon^2/4\tau_0^2}\,\mathrm{erfc}\left(\frac{t}{\varepsilon} + \frac{\varepsilon}{2\tau_0}\right). \tag{10}$$

This would be precisely the reduction constructed by the method of successive approximations. By using the properties of the complementary error function, it is easy to see that in the limit $\tau_0 \to 0$ one recovers the radiationless result (8), while for $\varepsilon \to 0$ one obtains the pre-accelerated solution (4). We see, thus, that these two limits do not commute.

In our opinion the delta function obtained in the limit $\varepsilon \to 0$ is beyond the field of applicability of the Lorentz–Dirac and Abraham–Lorentz equations, for which one has to assume that the applied force and acceleration do not change too much across a time interval of length τ_0, i.e. the radiation reaction cannot be too important along such a tiny interval. Since this assumption is not met by the delta function, this often useful limit is not applicable here and one has necessarily to consider pulses of width larger than τ_0. This opinion is in agreement with the point of view of the authors of [6–8] that stress that the analyticity with respect to τ_0 is a fundamental hypothesis, which is used in standard derivations of the Lorentz–Dirac equation.

3. An exact example

In [1] we discussed a linear one-dimensional exact example in the frame of the method of successive approximations. We now want to analyse a three-dimensional exact example that, though still linear, has a clearer physical meaning and will contribute to our confidence on the convergence of the method of successive approximations under appropriate conditions. Let us consider a charge e that moves in an external magnetic field $\boldsymbol{B}$, as happens in some astrophysical contexts [9] or in particle accelerators [10]. In the non-relativistic approximation the equation of motion is

$$\ddot{\boldsymbol{x}} = \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} + \tau_0 \dddot{\boldsymbol{x}} \tag{11}$$

where we have introduced the cyclotron frequency

$$\boldsymbol{\Omega} = -\frac{e\boldsymbol{B}}{m} \tag{12}$$

which we will assume to be uniform and constant. Starting from the lowest-order approximation

$$\ddot{\boldsymbol{x}} = \boldsymbol{\Theta}_0 \equiv \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} \tag{13}$$

we can construct successive approximations by using repeatedly

$$\ddot{\boldsymbol{x}} = \boldsymbol{\Theta}_{n+1} \equiv \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} + \tau_0 \left[\frac{\partial \boldsymbol{\Theta}_n}{\partial t} + (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{\Theta}_n + (\boldsymbol{\Theta}_n \cdot \boldsymbol{\nabla}_{\dot{\boldsymbol{x}}})\boldsymbol{\Theta}_n \right]. \tag{14}$$

It is straightforward to check that the successive approximations are

$$\boldsymbol{\Theta}_n = \alpha_n \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - \beta_n \dot{\boldsymbol{x}}_\perp \tag{15}$$

where

$$\dot{\boldsymbol{x}}_\perp \equiv \dot{\boldsymbol{x}} - \frac{\boldsymbol{\Omega} \cdot \dot{\boldsymbol{x}}}{\Omega^2} \boldsymbol{\Omega} \tag{16}$$

is the component of the velocity perpendicular to the magnetic field and the constant coefficients are given by the recurrence

$$\alpha_{n+1} = 1 - 2\tau_0 \alpha_n \beta_n \tag{17}$$

$$\beta_{n+1} = \tau_0 (\Omega^2 \alpha_n^2 - \beta_n^2) \tag{18}$$

and the initial conditions $\alpha_0 = 1$ and $\beta_0 = 0$.

Recurrence (18) has two fixed points $P_\pm = (\alpha_\pm, \beta_\pm)$ with

$$\alpha_\pm = \pm \frac{\sqrt{\frac{1}{2}\left(\sqrt{1+16\tau_0^2\Omega^2}-1\right)}}{2\tau_0\Omega} \tag{19}$$

$$\beta_\pm = \frac{\pm\sqrt{\frac{1}{2}\left(\sqrt{1+16\tau_0^2\Omega^2}+1\right)}-1}{2\tau_0} \tag{20}$$

but a linear stability analysis proves that P_- is always unstable and that P_+ is asymptotically stable for

$$\tau_0\Omega < \frac{\sqrt{3+2\sqrt{3}}}{4} \approx 0.64. \tag{21}$$

Furthermore, a simple bifurcation diagram in the dimensionless variables $(\alpha_n, \tau_0\beta_n)$ shows that the initial condition $(1, 0)$ is in the basin of attraction of P_+ and, as a consequence, that the method of successive approximations will in fact converge in the range (21) to the Newtonian equation

$$\ddot{\boldsymbol{x}} = \alpha_+ \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - \beta_+ \dot{\boldsymbol{x}}_\perp \tag{22}$$

which contains precisely the physical (non-runaway) solutions for $\boldsymbol{x}$ found by Plass [5]. Notice that all the approximations (15), as well as the exact order reduction (22), are orthogonal to the magnetic field, and that the exact reduction exists even when the method fails. This is not surprising because most approximation methods have limited ranges of applicability. Moreover, in this case the method will converge in all practical situations because the cyclotron frequency is always very small compared to $1/\tau_0$. This simple but illustrative exact example reinforces our conviction that the numerical approximation method [1] will converge in many cases of interest.

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References

[1] Aguirregabiria J M 1997 *J. Phys. A: Math. Gen.* **30** 2391
[2] Rohrlich F 1965 *Classical Charged Particles* (Reading, MA: Addison-Wesley)
[3] Bel Ll and Sirousse-Zia H 1985 *Phys. Rev.* D **32** 3128
[4] Parker L and Simon J Z 1993 *Phys. Rev.* D **47** 1339
[5] Plass N G 1961 *Rev. Mod. Phys.* **33** 37
[6] Bhabha H J 1946 *Phys. Rev.* **70** 759
[7] Valentini A 1988 *Phys. Rev. Lett.* **61** 1903
[8] Flanagan É É and Wald R M 1996 *Phys. Rev.* D **54** 6233
[9] Ginzburg V L 1989 *Applications of Electrodynamics in Theoretical Physics and Astrophysics* (New York: Gordon and Breach)
[10] Schwinger J 1949 *Phys. Rev.* **75** 1912